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Global solutions to the Boltzmann equation near equilibrium in the Besov spaces

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1 Introduction

We shall consider a Cauchy problem of the Boltzmann equation

$$\begin{cases} \partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = Q(F,F)(t,x,v), \\ F(0,x,v) = F_0(x,v), \end{cases} \tag{1}$$

where $t > 0$ is time, $v \in \mathbb{R}^3$ is velocity of a particle, $x \in \mathbb{R}^3$ is position of a particle, and the unknown $f = f(t,x,v)$ is a probability density function of a dilute gas. The Boltzmann equation is a model for such a gas where interactions of particles consisting of the gas play important roles in formulation of phenomena. The collision operator Q is a bilinear form written

$$Q(F,G)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\omega)(F(v_*)'G(v') - F(v_*)G(v))\, d\omega dv_*.$$

The velocity variables v, v_*, v' and v'_* satisfies the relations

$$v' = v - (v-v_*)\cdot\omega\omega, \quad v'_* = v_* + (v-v_*)\cdot\omega\omega$$

for $\omega \in \mathbb{S}^2$, which are equivalent to the conservation laws of momentum and energy in collisions

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2. \tag{2}$$

As a physically suitable model, we assume that the collision kernel B is a product of two functions, namely

$$B(v-v_*,\omega) = |v-v_*|^\gamma b(\cos\theta),$$

where $-3 < \gamma \le 1$. We assume that $0 \le b(\cos\theta) \le C|\cos\theta|$ for some $C > 0$. This is called the cutoff assumption (we may assume boundedness of integration of b over $\mathbb{S}^2$, which includes some singularity of b, when we say the cutoff assumption, but our assumption is enough for general arguments).

It is well-known that the Maxwell distribution (often referred as the Maxwellian)

$$M = M(v) := \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}$$

is an equiliburium of (1), which can be easily verified by (2). Hence we substitute $F = M + M^{1/2}f$ into (1) and reformulate as an equation of f; then it reads

$$\begin{cases}\partial_t f + v\cdot\nabla_x f + Lf = \Gamma(f,f),\\ f(0,x,v) = f_0(x,v) := M^{-1/2}[F_0(x,v) - M].\end{cases} \tag{3}$$

Here L and Γ are the linear and the nonlinear part of Q, respectively, that is,

$$\begin{aligned}&Lf = -M^{-1/2}[Q(M, M^{1/2}f) + Q(M^{1/2}f, M)],\\ &\Gamma(f,g) = M^{-1/2}Q(M^{1/2}f, M^{1/2}g).\end{aligned}$$

Under the cutoff assumption, it is known that L can be decomposed into $\nu - K$, where ν is a multiplicator equivalent to $\langle v\rangle^\gamma$, and K is an integral operator.

Our aim is to establish existence and uniqueness of a solution to (3) in the Besov spaces with respect to the space variable x, especially for the soft potential case $-3 < \gamma < 0$. Also, it is our aim to compare the results by Duan and the author [8] with those in [7] and [12] so that we can see in what direction this research is going on.

Before stating our main theorem, we show the preceding results for comparison with ours. For that purpose, we introduce the macro-micro decomposition. It is known that L is nonnegative-definite on $L^2_v(\mathbb{R}^3)$ and

$$\ker L = \operatorname{span}\{\mu^{1/2}, \mu^{1/2}v_i\ (i = 1,2,3), \mu^{1/2}|v|^2\}.$$

$\mathbf{P}$ denotes the projection operator onto $\ker L$; therefore for each f, there exist functions a, b, c such that it holds

$$\mathbf{P}f(x,v,t) = \left[a(x,t) + v\cdot b(x,t) + |v|^2 c(x,t)\right]\mu^{1/2}(v).$$

The decomposition $f = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f$ is called the macro-micro decomposition.

For definitions of these norms, we refer Section 2. We define the energy term and the dissipation term by

$$\mathcal{E}_T(f) \sim \|f\|_{\tilde L^\infty_T \tilde L^2_v(B^{3/2}_{2,1})}, \quad \mathcal{D}_T(f) = \|\nabla_x(a,b,c)\|_{\tilde L^2_T(B^{1/2}_{2,1})} + \|(\mathbf{I}-\mathbf{P})f\|_{\tilde L^2_T \tilde L^2_{v,\nu}(B^{3/2}_{2,1})}$$

for the first result giving the unique solution near Maxwellian in the Besov space.

Theorem 1.1. *([7]) There are $\varepsilon_0 > 0$, $C > 0$ such that if*

$$\|f_0\|_{\tilde L^2_v(B^{3/2}_{2,1})} \le \varepsilon_0,$$

then there exists a unique global strong solution $f(x,v,t)$ to (3) *with initial data $f_0(x,v)$, satisfying*

$$\mathcal{E}_T(f) + \mathcal{D}_T(f) \le C\|f_0\|_{\tilde L^2_v(B^{3/2}_{2,1})}$$

for any $T > 0$.

Novelty of this result is in showing well-posedness of (3) in the Besov space, and the differentiability index s can take the value 3/2, which is the best to our knowledge.

Readers interested in preceding well-posedness results in the Sobolev spaces may refer to [7], which extensively collects research on well-posedness of the Boltzmann equation near Maxwellian. We mention that this result is generalized by Tang and Liu [15] in view of indices under the same assumption. These results concern with the cutoff case. In order to apply techniques developed in [7] to the non-cutoff case, Morimoto and the author [12] conbined these with those developed in a series of AMUXY papers, such as [1, 2, 3]. They obtained the following result. Replace the energy and dissipation terms with

$$E_T(f) \sim \|f\|_{\tilde{L}^\infty_T \tilde{L}^2_v(B^{3/2}_{2,1})}, \quad D_T(f) = \|\nabla_x(a,b,c)\|_{\tilde{L}^2_T(B^{1/2}_{2,1})} + \|(\mathbf{I}-\mathbf{P})f\|_{\mathcal{T}^{3/2}_{T,2,2}}.$$

Theorem 1.2. *([12]) Let* $0 < \mathsf{s} < 1$ *and* $\gamma > \max\{-3, -2\mathsf{s} - 3/2\}$*. Instead of the cutoff assumption* $0 \le b(\cos\theta) \le C|\cos\theta|$*, we assume* $b(\cos\theta) \sim \theta^{-2-2\mathsf{s}}$ *as* $\theta \downarrow 0$*. Then there are constants* $\varepsilon_0 > 0$ *and* $C > 0$ *such that if*

$$\|f_0\|_{\tilde{L}^2_v(B^{3/2}_{2,1})} \le \varepsilon_0,$$

then there exists a unique global solution $f(x,v,t)$ *of* (3) *with initial datum* $f_0(x,v)$*. This solution satisfies*

$$E_T(f) + D_T(f) \le C\|f_0\|_{\tilde{L}^2_v(B^{3/2}_{2,1})}$$

Positivity of the solutions is also shown both in [7] and [12]. Novelty of Theorem 1.2 is that they treated the non-cutoff case, which is much harder to tackle than the cutoff case, and they succeeded in proving the same solution space used in [7] applies to their problem. This yields the smallest differentiability index 3/2 for the non-cutoff case, better than those obtained in [1, 2] or [10]. We remark that in both results, the case γ is very close to -3, which is sometimes called the very soft potential case, cannot be handled. It is the main motivation of [8] to establish a solution in Besov spaces for that case.

We shall state the main theorems of [8]. In order to do so, we first clarify in what sense $f(t,x,v)$ is a solution to (3). In fact, the mild solution $f(t,x,v)$ to (3) is defined as the following integral form:

$$\begin{aligned} f(t,x,v) =& e^{-\nu(v)t} f_0(x - vt, v) + \int_0^t e^{-\nu(v)(t-s)}(Kf)(s, x-(t-s)v, v)\,ds \\ &+ \int_0^t e^{-\nu(v)(t-s)}\Gamma(f,f)(s, x-(t-s)v, v)\,ds, \end{aligned}$$

for $t \ge 0$, x, $v \in \mathbb{R}^3$. In what follows, for a Banach space X and a nonnegative constant $\alpha \ge 0$ we define

$$|\!|\!| f |\!|\!|_{\alpha,X} = \sup_{t\ge 0}(1+t)^\alpha \|f(t)\|_X,$$

for a X-valued function $f(t)$ on the real half line $0 \le t < \infty$, and for any Banach spaces X and Y, the norm $\|\cdot\|_{X\cap Y}$ means $\|\cdot\|_X + \|\cdot\|_Y$. For more notations of function spaces, especially Besov and Chemin-Lerner type spaces, readers may refer to the next preliminary section.

The following are the results for the hard and the soft potential cases, respectively.

Theorem 1.3. *Assume $0 \le \gamma \le 1$, $q \in [1,2]$, $s \ge 3/2$, and $\beta > \gamma + 3/2$. Then there exist positive constants $\varepsilon > 0$ and $C > 0$ such that if initial data f_0 satisfies*

$$\|f_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1}) \cap L^2_v L^q_x} \le \varepsilon,$$

then the Cauchy problem (3) *admits a unique global mild solution*

$$f(t,x,v) \in L^\infty(0,\infty; \tilde{L}^\infty_\beta(B^s_{2,1}))$$

satisfying

$$\|\!| f \|\!|_{\alpha, \tilde{L}^\infty_\beta(B^s_{2,1})} \le C \|f_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1}) \cap L^2_v L^q_x},$$

where $\alpha = d/2(1/q - 1/2)$.

Theorem 1.4. *Assume $-3 < \gamma < 0$, $s \ge 3/2$, $\sigma = 3|\gamma|/4$, and $\beta > \sigma_+ + 3/2$, where σ_+ denotes $\sigma + \delta$ for an arbitrary small constant $\delta > 0$. Then there exist positive constants $\varepsilon > 0$ and $C > 0$ such that if initial data f_0 satisfies*

$$\|f_0\|_{\tilde{L}^\infty_{\beta+\sigma}(B^s_{2,1}) \cap L^2_{\sigma_+} L^1_x} \le \varepsilon,$$

then the Cauchy problem (3) *admits a unique global mild solution*

$$f(t,x,v) \in L^\infty(0,\infty; \tilde{L}^\infty_\beta(B^s_{2,1}))$$

satisfying

$$\|\!| f \|\!|_{3/4, \tilde{L}^\infty_\beta(B^s_{2,1})} \le C \|f_0\|_{\tilde{L}^\infty_{\beta+\sigma}(B^s_{2,1}) \cap L^2_{\sigma_+} L^1_x}.$$

Here we only gave the results for the 3 dimensional case for comparison with [7] and [12], which treat such the case, but these can be generalized into the d dimensional case ($d \ge 1$ for Theorem 1.3 and $d \ge 3$ for Theorem 1.4). See [8] for finer details.

In this article, we will give outlined proof of Theorem 1.4 in more explanatory way than that given in [8]. We give preliminary facts and lemmas in Section 2, and give outlined proof of Theorem 1.4 in Section 3.

2 Preliminaries

Following [5], we define the Besov spaces in this section. We first introduce the Littlewood-Paley decomposition. Let $\mathcal{A}$ be the annulus $\{\xi \in \mathbb{R}^3 \mid 3/4 \le |\xi| \le 8/3\}$ and $\mathcal{B}$ be the ball $B(0, 4/3)$. Then there exist radial functions $\chi \in C_0^\infty(\mathcal{A})$ and $\phi \in C_0^\infty(\mathcal{C})$ satisfying the following:

$$0 \le \chi(\xi) \le 1 \text{ and } 0 \le \phi(\xi) \le 1 \text{ for any } \xi \in \mathbb{R}^3,$$
$$\chi(\xi) + \sum_{q \ge 0} \phi(2^{-q}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^3,$$
$$\sum_{q \in \mathbb{Z}} \phi(2^{-q}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^3 \setminus \{0\},$$
$$\operatorname{supp} \phi(2^{-q}\,\cdot) \cap \operatorname{supp} \phi(2^{-q'}\cdot) = \emptyset \text{ for any } q, q' \text{ with } |q - q'| \ge 2,$$
$$\operatorname{supp} \chi \cap \operatorname{supp} \phi(2^{-q}\,\cdot) = \emptyset \text{ for any } q \ge 1, \text{ and}$$
$$2^{q'}\tilde{\mathcal{A}} \cap 2^q\mathcal{A} = \emptyset \text{ for any } q, q' \text{ with } |q - q'| \ge 5,$$

where $\tilde{\mathcal{A}} := B(0, 2/3) + \mathcal{A}$. We write $h := \mathcal{F}^{-1}\phi$ and $\tilde{h} := \mathcal{F}^{-1}\chi$. For each $f \in \mathcal{S}'(\mathbb{R}^3_x)$, the non-homogeneous dyadic blocks Δ_q are defined by

$$\Delta_{-1} f := \chi(D) f = \int_{\mathbb{R}^3} \tilde{h}(y) f(x - y) dy,$$
$$\Delta_q f := \phi(2^{-q}D) f = 2^{3q} \int_{\mathbb{R}^3} h(2^q y) f(x - y) dy \quad (q \in \mathbb{N} \cup \{0\})$$

and $\Delta_q f := 0$ if $q \le -2$. The non-homogeneous low-frequency cutoff operator S_q is defined by

$$S_q f = \sum_{q' \le q-1} \Delta_{q'} f.$$

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^3)$ is the usual L^p-space endowed with $\|\cdot\|_{L^p}$ For $1 \le p$, $q \le \infty$, we define

$$L^p_v L^q_x = L^p\left(\mathbb{R}^3_v; L^q(\mathbb{R}^3_x)\right), \quad L^q_x L^p_v = L^q\left(\mathbb{R}^3_x; L^p(\mathbb{R}^3_v)\right).$$

A velocity-weighted L^p space with a weight index $\beta \in \mathbb{R}$ is defined as

$$L^p_\beta = \left\{ f = f(v) \,\middle|\, \langle \cdot \rangle^\beta f \in L^p \right\}, \quad \|f\|_{L^p_\beta} := \|\langle \cdot \rangle^\beta f\|_{L^p}$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$.

We denote the set of all polynomials on $\mathbb{R}^3$ by $\mathcal{P}$. The homogeneous dyadic blocks $\dot{\Delta}_q$ are defined by

$$\dot{\Delta}_q f := \phi(2^{-q}D) f := 2^{3q} \int_{\mathbb{R}^3} h(2^q y) f(x - y) dy$$

for any $f \in \mathcal{S}'(\mathbb{R}^3_x)/\mathcal{P}$ and $q \in \mathbb{Z}$. Since it holds that $\dot{\Delta}_q P = 0$ for any $P \in \mathcal{P} \subset \mathcal{S}'$ and $q \in \mathbb{Z}$, it is reasonable to define $\dot{\Delta}_q$ over the quotient space.

We are now in position to give the definition of non-homogeneous Besov spaces.

Definition 2.1. Let $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. The nonhomogeneous Besov space B^s_{pr} is defined by

$$B^s_{pr} := \left\{ f \in \mathcal{S}'(\mathbb{R}^3_x) \mid \|f\|_{B^s_{pr}} := \|(2^{qs}\|\Delta_q f\|_{L^p_x})_{q \ge -1}\|_{\ell^r} < \infty \right\}$$

with the obvious modification for the case $r = \infty$.

The definition of homogeneous Besov spaces is as follows:

Definition 2.2. Let $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. The homogeneous Besov space $\dot{B}^s_{pr}$ is defined by

$$\dot{B}^s_{pr} := \left\{ f \in \mathcal{S}'(\mathbb{R}^3_x)/\mathcal{P} \mid \|f\|_{\dot{B}^s_{pr}} := \left\| (2^{qs} \|\dot{\Delta}_q f\|_{L^p_x})_{q\in\mathbb{Z}} \right\|_{\ell^r} < \infty \right\}$$

with the obvious modification for the case $r = \infty$.

Next, we define Chemin-Lerner spaces, which is (to some extent) a generalization of the Besov space.

Definition 2.3. Let $1 \le p, r, \alpha, \beta \le \infty$ and $s \in \mathbb{R}$.

$$\tilde{L}^2_v(B^s_{2,1}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d_x \times \mathbb{R}^d_v) \;\middle|\; \|f\|_{\tilde{L}^2_v(B^s_{2,1})} = \overline{\sum} \|\Delta_j f\|_{L^2_v L^2_x} < \infty \right\},$$
$$\tilde{L}^\infty_\beta(B^s_{2,1}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d_x \times \mathbb{R}^d_v) \;\middle|\; \|f\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} = \overline{\sum} \sup_v \langle v\rangle^\beta \|\Delta_j f(\cdot, v)\|_{L^2_x} < \infty \right\}.$$

For $T \in [0, \infty)$, the Chemin-Lerner space $\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr})$ is defined by

$$\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr}) := \left\{ f(\cdot, v, t) \in \mathcal{S}' \mid \|f\|_{\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr})} < \infty \right\},$$

where

$$\|f\|_{\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr})} := \left\| (2^{qs} \|\Delta_q f\|_{L^\alpha_T L^\beta_v L^p_x})_{q\ge -1} \right\|_{\ell^r},$$
$$\|\Delta_q f\|_{L^\alpha_T L^\beta_v L^p_x} := \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\Delta_q f(x, v, t)|^p dx \right)^{\beta/p} dv \right)^{\alpha/\beta} dt \right)^{1/\alpha}$$

with the usual convention when at least one of p, r, α, β is equal to ∞. We also define $\tilde{L}^\alpha_T \tilde{L}^\beta_v(\dot{B}^s_{pr})$ similarly.

We denote $\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{2,1})$ by $\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_x)$, and $\tilde{L}^\alpha_T \tilde{L}^\beta_v(\dot{B}^s_{2,1})$ by $\tilde{L}^\alpha_T \tilde{L}^\beta_v(\dot{B}^s_x)$. The spaces $\tilde{L}^2_v(B^s_{2,1})$ and $\tilde{L}^\infty_\beta(B^s_{2,1})$ play essential role in [8], while $\tilde{L}^\infty_T \tilde{L}^2_v(B^{3/2}_{2,1})$, $\tilde{L}^2_T \tilde{L}^2_{v,\nu}(B^{3/2}_{2,1})$ and $\mathcal{T}^{3/2}_{T,2,2}$ were employed in [7] and [12].

Finally, we give the definition of the non-isotropic norm $|\!|\!| \cdot |\!|\!|$ and the space $\mathcal{T}^s_{Tpr}$ and $\dot{\mathcal{T}}^s_{Tpr}$, which are endowed with the 'Chemin-Lerner type triple norm'. This is used in [12] for the dissipation term to take advantages of AMUXY's works.

Definition 2.4. Let $1 \le p, r \le \infty$, $T > 0$ and $s \in \mathbb{R}$. $|\!|\!| f |\!|\!|$ is defined by

$$\begin{aligned} |\!|\!| f |\!|\!|^2 := & \iiint B(v - v_*, \sigma)\mu_*(f' - f)^2 dv dv_* d\sigma \\ & + \iiint B(v - v_*, \sigma) f_*^2 (\sqrt{\mu'} - \sqrt{\mu})^2 dv dv_* d\sigma \\ = & J_1^{\Phi_\gamma}(g) + J_2^{\Phi_\gamma}(g), \end{aligned}$$

and the space $\mathcal{T}^s_{Tpr}$ is defined by

$$\mathcal{T}^s_{Tpr} := \left\{ f \mid \|f\|_{\mathcal{T}^s_{Tpr}} = \left\| (2^{qs} \| |\!|\!| \Delta_q f |\!|\!| \|_{L^p_T L^r_x})_{q \geq -1} \right\|_{\ell^1} < \infty \right\}.$$

$\dot{\mathcal{T}}^s_{Tpr}$ is defined in the same manner.

$|\!|\!| \cdot |\!|\!|$ is called the triple norm, and it is known that this norm is estimated from both above and below by weighted Sobolev norms:

$$\|f\|^2_{H^{\nu/2}_{\gamma/2}} + \|f\|^2_{L^2_{(\nu+\gamma)/2}} \lesssim |\!|\!| f |\!|\!|^2 \lesssim \|f\|^2_{H^{\nu/2}_{(\nu+\gamma)/2}}.$$

In order to deduce Chemin-Lerner estimates in the following sections, we will use some properties of the above spaces. Here we give some of them.

Lemma 2.5. *There exists a constant $C > 0$ such that for any $1 \leq p \leq \infty$ and $f \in L^p_x$ it holds*

$$\|\Delta_q f\|_{L^p_x} \leq C\|f\|_{L^p_x}, \quad \|S_q f\|_{L^p_x} \leq C\|f\|_{L^p_x}.$$

In short, Δ_q and S_q are bounded operators on L^p_x.

Lemma 2.6. *Let $1 \leq p, r \leq \infty$. Then*

1. *$B^{s_1}_{pr} \hookrightarrow B^{s_2}_{pr}$ when $s_2 \leq s_1$. This inclusion does not hold for the homogeneous case.*
2. *$B^{3/p}_{p,1} \hookrightarrow L^\infty$ and $\dot{B}^{3/p}_{p,1} \hookrightarrow L^\infty$ when $1 \leq p < \infty$.*

Lemma 2.7. *Let $1 \leq p, q, r \leq \infty$ and $s > 0$. Then we have*

$$\|\nabla_x f\|_{\tilde{L}^q_T(\dot{B}^s_{pr})} \sim \|f\|_{\tilde{L}^q_T(\dot{B}^{s+1}_{pr})}, \quad \|f\|_{\tilde{L}^q_T(\dot{B}^s_{pr})} \lesssim \|f\|_{\tilde{L}^q_T(B^s_{pr})}.$$

Lemma 2.8. *Let $1 \leq p, \alpha, \beta, r \leq \infty$ and $s > 0$. If $r \leq \min\{\alpha, \beta\}$ then*

$$\|f\|_{L^\alpha_T L^\beta_v(B^s_{pr})} \leq \|f\|_{\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr})} \text{ and } \|f\|_{L^\alpha_T L^\beta_v(\dot{B}^s_{pr})} \leq \|f\|_{\tilde{L}^\alpha_T \tilde{L}^\beta_v(\dot{B}^s_{pr})}.$$

We emphasize that $\|\cdot\|_{\tilde{L}^\alpha_T \tilde{L}^\beta_v(B^s_{pr})}$ is usually easier to handle than $\|\cdot\|_{L^\alpha_T L^\beta_v(B^s_{pr})}$, because summation comes after all the integration in definition of the former one. The above Chemin-Lerner spaces are used in [7] for the cutoff case and in [12] for the non-cutoff case.

Regarding the time-decay property in the soft potential case, we cite the following lemma from [9] (see also [14]). Note that, compared to [16], which treats the case $-1 < \gamma$ for technical reason, γ can take the full range of values for soft potentials.

Lemma 2.9. *Let $-3 < \gamma < 0$, and let $\ell \geq 0$, $J > 0$ be given constants. Set $\mu = \mu(v) := \langle v \rangle^{-\gamma/2}$. There is a nonnegative time-frequency functional $\mathcal{E}_\ell(t, \xi) = \mathcal{E}_\ell(\hat{f}(t, \xi))$ with*

$$\mathcal{E}_\ell(t, \xi) \sim \|\mu^\ell \hat{f}(t, \xi)\|^2_{L^2_v}$$

such that the solution to the Cauchy problem on the linearized homogeneous equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + Lf = 0, \\ f(0, x, v) = f_0(x, v) \end{cases} \tag{4}$$

satisfies

$$\mathcal{E}_\ell(t, \xi) \leq C(1 + \rho(\xi)t)^{-J}\mathcal{E}_{\ell+J_+}(0, \xi),$$

for all $t \geq 0$ *and* $\xi \in \mathbb{R}^d$*, where* $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$*, and* $C > 0$ *is a generic constant.*

3 Proof of Theorem 1.4

We now turn to the proof of Theorem 1.4. We start from considering the time-decay in the space $\tilde{L}^2_v(B^s_{2,1})$ for the solution to the Cauchy problem (4) with the help of Lemma 2.9 whose proof is based on the pure energy method.

Lemma 3.1. *Assume* $-3 < \gamma < 0$*. Take* $\ell \geq 0$*,* $1 \leq q \leq 2$*, and* $J > 3(1/q - 1/2) = 2\alpha$*. Let* $f(t, x, v)$ *be the solution to the Cauchy problem* (4) *with initial data* $f_0(x, v)$*. Then it holds that*

$$\begin{aligned} \|\nu^{-\ell/2} f(t)\|_{\tilde{L}^2_v(B^s_{2,1})} \leq & C(1+t)^{-J/2}\|\nu^{-(\ell+J_+)/2} f_0\|_{\tilde{L}^2_v(B^s_{2,1})} \\ & + C(1+t)^{-\alpha}\|\nu^{-(\ell+J_+)/2} f_0\|_{L^2_v L^q_x} \end{aligned} \tag{5}$$

for all $t \geq 0$*.*

Proof. Recall $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$. The proof is based on the usual decomposition into low- and high-frequency parts in the Fourier variable. By Lemma 2.9 we have

$$\begin{aligned} \|\nu^{-\ell/2}\Delta_j f(t)\|^2_{L^2_{x.v}} &= \int_{\mathbb{R}^d} \|\nu^{-\ell/2}\phi(2^{-j}\xi)\hat{f}(t, \xi)\|^2_{L^2_v} d\xi \\ &\leq C\int_{\mathbb{R}^d} (1 + \rho(\xi)t)^{-J}\|\nu^{-(\ell+J_+)/2}\phi(2^{-j}\xi)\hat{f}_0(\xi)\|^2_{L^2_v} d\xi \\ &= C\left\{\int_{|\xi|\geq 1} + \int_{|\xi|\leq 1}\right\}(\cdots)d\xi =: I_1 + I_2. \end{aligned}$$

For the high-frequency part I_1, we notice $1 + \rho(\xi)t \sim 1 + t$ on $\{|\xi| \geq 1\}$. Thus, one has

$$I_1 \leq C(1+t)^{-J}\|\nu^{-(\ell+J_+)/2}\Delta_j f_0\|^2_{L^2_{x,v}}.$$

For the low-frequency part I_2, we take the triplet (q, p, p') satisfying $1/p + 1/p' = 1$ and $1/2p + 1/q = 1$, where q is given in the assumption. The Hölder inequality gives

$$\begin{aligned} I_2 &\leq \Big(\int_{|\xi|\leq 1} (1 + \rho(\xi)t)^{-Jp'}d\xi\Big)^{1/p'}\Big(\int_{|\xi|\leq 1} \|\nu^{-(\ell+J_+)/2}\phi(2^{-j}\xi)\hat{f}_0(\xi)\|^{2p}_{L^2_v} d\xi\Big)^{1/p} \\ &\leq C\tilde{\psi}_{Jp'}(t)^{1/p'}\Big(\int_{|\xi|\leq 1} \|\nu^{-(\ell+J_+)/2}\phi(2^{-j}\xi)\hat{f}_0(\xi)\|^{2p}_{L^2_v} d\xi\Big)^{1/p}, \end{aligned}$$

where

$$\tilde{\psi}_{Jp'}(t) = \int_0^1 \Big(1 + \frac{r^2}{1+r^2}t\Big)^{-Jp'} r^2 dr.$$

By the change of variable $r \to s = r^2 t/(1+r^2)$, it holds that

$$\tilde{\psi}_{Jp'}(t) = \int_0^{t/2} (1+s)^{-Jp'} \Big(\frac{s}{t-s}\Big)\frac{t}{2\sqrt{s}}(t-s)^{-3/2} ds \le C(1+t)^{-3/2},$$

due to $J > 3(1/q - 1/2) = 3/2p'$. Therefore, combining estimates on I_1 and I_2, we have obtained

$$\begin{aligned}\|\nu^{-\ell/2}\Delta_j f(t)\|^2_{L^2_{x.v}} &\le C(1+t)^{-J}\|\nu^{-(\ell+J_+)/2}\Delta_j f_0\|^2_{L^2_{x,v}} \\ &\quad + C(1+t)^{-2\alpha}\Big(\int_{|\xi|\le 1} \|\nu^{-(\ell+J_+)/2}\phi(2^{-j}\xi)\hat{f}_0(\xi)\|^{2p}_{L^2_v} d\xi\Big)^{1/p}.\end{aligned}$$

Taking the square root of the above inequality, further taking summation with respect to j with the weight 2^{js}, and noticing that

$$\overline{\sum}\Big(\int_{|\xi|\le 1} \|\nu^{-(\ell+J_+)/2}\phi(2^{-j}\xi)\hat{f}_0(\xi)\|^{2p}_{L^2_v} d\xi\Big)^{1/2p}$$

is bounded by $C\|\nu^{-(\ell+J_+)/2} f_0\|_{L^2_v L^q_x}$, the desired estimate (5) then follows. This completes the proof. $\square$

In [7] and [12], the macro-micro decomposition is fully used to derive estimates of the energy and dissipative terms, which close a priori estimates. On the other hand, in [8] we are looking for a mild solution which require different recipe, and by Lemma 3.1 one can further derive the time-decay of solutions in the space $\tilde{L}^\infty_v(B^s_{2,1})$ with a suitable velocity weight.

Lemma 3.2. *Assume $-3 < \gamma < 0$, $0 < \beta < |\gamma| + 2$, and $\beta' \ge 0$. Then the solution $f(t,x,v)$ to the Cauchy problem* (4) *with initial data $f_0(x,v)$ satisfies*

$$\|f(t)\|_{\tilde{L}^\infty_{\beta'}(B^s_{2,1})} \le C(1+t)^{\beta/\gamma}\left(\|f_0\|_{\tilde{L}^\infty_{\beta+\beta'}(B^s_{2,1})} + \|\!|f|\!\|_{\beta/|\gamma|,\tilde{L}^2_v(B^s_{2,1})}\right), \tag{6}$$

for all $t \ge 0$.

Outline of proof. We shall follow the proof of [6, Lemma 4.5]. First, due to $L = \nu - K$, we write the linearized equation in the form of

$$\partial_t f + v\cdot\nabla_x f + \nu f = Kf.$$

Define $h(t,x,v) = \langle v\rangle^{\beta'} f(t,x,v)$. Then the equation for h reads

$$\partial_t h + v\cdot\nabla_x h + \nu h = K_{\beta'} h,$$

where we have defined

$$K_{\beta'}(h)(v) = \langle v\rangle^{\beta'}\ K\Big(\frac{h}{\langle\cdot\rangle^{\beta'}}\Big)(v) = \int_{\mathbb{R}^d} k_{\beta'}(v,v')h(v')dv',$$

with a new integral kernel $k_{\beta'}(v,v') = k(v,v')\langle v\rangle^{\beta'}/\langle v'\rangle^{\beta'}$. Therefore, to show the desired estimate (6) it suffices to prove

$$\|h(t)\|_{\tilde{L}^\infty_0(B^s_{2,1})} \le C(1+t)^{\beta/\gamma}\Big(\|h_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} + \|\!|f|\!\|_{\beta/|\gamma|,\tilde{L}^2_v(B^s_{2,1})}\Big), \tag{7}$$

for all $t \ge 0$. Indeed, the mild form of the equation for h is written as

$$h(t,x,v) = e^{-\nu(v)t}h_0(x-vt,v) + \int_0^t e^{-\nu(v)(t-s)}(K^m_{\beta'} + K^c_{\beta'})h(s,x-(t-s)v,v)ds, \tag{8}$$

where we have denoted

$$(K^m_{\beta'}h)(v) = \int k^m_{\beta'}(v,v_*)\chi_m(|v-v_*|)h(v_*)dv_*$$

with

$$0 \le \chi_m \le 1, \chi_m(t) = 1 \text{ for } t \le m,\ \ \chi_m(t) = 0 \text{ for } t \ge 2m,$$

and $K^c_{\beta'} = K_{\beta'} - K^m_{\beta'}$. The small constant $m > 0$ will be chosen later. Applying Δ_j to (8) and taking the L^2_x-norm, we have

$$\begin{aligned}\|\Delta_j h(t,v)\|_{L^2_x} \le{}& e^{-\nu(v)t}\|\Delta_j h_0(v)\|_{L^2_x} + \int_0^t \|\Delta_j(K^m_{\beta'}h)(s,v)\|_{L^2_x}ds\\ &+ \int_0^t \|\Delta_j(K^c_{\beta'}h)(s,v)\|_{L^2_x}ds\\ =:{}& L^j_1 + L^j_2 + L^j_3.\end{aligned} \tag{9}$$

To the end, for brevity we put $\tilde{\alpha} = \beta/|\gamma| > 0$ and

$$\|\!|h|\!\| = \|\!|h|\!\|_{\beta/|\gamma|,\tilde{L}^\infty_v(B^s_{2,1})} = \|\!|h|\!\|_{\tilde{\alpha},\tilde{L}^\infty_v(B^s_{2,1})}.$$

Not to be confused with the triple norm. Notice $0 < \tilde{\alpha} < 1 - 2/\gamma$.

Before starting the estimates on L^j_k $(k = 1,2,3)$ in (9), we recall some useful facts for $K^m_{\beta'}$ and $K^c_{\beta'}$, cf. [6].

$$|(K^m_{\beta'}g)(v)| \le Cm^{d+\gamma}e^{-|v|^2/10}\|g\|_{L^\infty}, \tag{10}$$

$$(K^c_{\beta'}g)(v) = \int_{\mathbb{R}^d} \ell^c_{\beta'}(v,\eta)g(\eta)d\eta,$$

$$\int_{\mathbb{R}^d} |\ell^c_{\beta'}(v,\eta)|d\eta \le C_\gamma m^{\gamma-1}\frac{\nu(v)}{1+|v|^2}, \tag{11}$$

$$\int_{\mathbb{R}^d} |\ell^c_{\beta'}(v,\eta)|e^{-|\eta|^2/20}d\eta \le Ce^{-|v|^2/100}. \tag{12}$$

Now, since it holds that $x^a e^{-x} \le C_a$ on $\{x \ge 0\}$ for each $a \ge 0$, we have

$$\overline{\sum} L_1^j \le C \overline{\sum} e^{-\langle v\rangle^\gamma t} (\langle v\rangle^\gamma t)^{\tilde{\alpha}}\ t^{-\tilde{\alpha}} \|\Delta_j h_0(v)\|_{L^\infty_\beta L^2_x} \le C(1+t)^{-\tilde{\alpha}} \|h_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1})}.$$

By (10) it holds that

$$|(K^m_{\beta'}1)(v)| \le C m^{d+\gamma} e^{-|v|^2/10} \langle v\rangle^{\beta'} \le C_{\beta'} m^{d+\gamma} e^{-|v|^2/15}.$$

Thus we have

$$\begin{aligned}
\overline{\sum} L_2^j &\le C_{\beta'} m^{d+\gamma} e^{-|v|^2/15} \|h\| \int_0^t e^{-\nu(v)(t-s)} (1+s)^{-\tilde{\alpha}} ds \\
&\le C_{\beta'} m^{d+\gamma} e^{-|v|^2/20} \|h\| \int_0^t (1+t-s)^{-\tilde{\alpha}-1} (1+s)^{-\tilde{\alpha}} ds \\
&\le C_{\beta'} m^{d+\gamma} e^{-|v|^2/20} \|h\| (1+t)^{-\tilde{\alpha}},
\end{aligned}$$

where we have used the inequality $e^{-|v|^2/10} e^{-\nu(v)(t-s)} \le C_b e^{-|v|^2/20} (1+t-s)^{-b}$ for $b \ge 0$. This then completes the estimates on L_1^j and L_2^j. Furthermore, by substituting those estimates into L_3^j, one has

$$\begin{aligned}
L_3^j \le & \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^d} |\ell^c_{\beta'}(v,v')| \Big[e^{-\nu(v')s} \|\Delta_j h_0(v')\|_{L^2_x} \\
& \qquad + \int_0^s e^{-\nu(v')(s-\tau)} \|\Delta_j (K^m_{\beta'} h)(\tau,v')\|_{L^2_x} d\tau \Big] dv' ds \\
& + \int_0^t e^{-\nu(v)(t-s)} \int_0^s e^{-\nu(v')(s-\tau)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |\ell^c_{\beta'}(v,v')\ell^c_{\beta'}(v',v'')| \\
& \qquad \times \|\Delta_j h(\tau,v'')\|_{L^2_x} dv'' dv' d\tau ds \\
=: & \ L_{31}^j + L_{32}^j + L_{33}^j.
\end{aligned}$$

It should be emphasized that, by this substitution, just one iteration is sufficient for proof. This is the novelty of methods given in [6] (see also [11]). L_{31}^j and L_{32}^j can be similarly estimated as L_1^j and L_2^j, respectively. In fact, it follows from (11) that

$$\overline{\sum} L_{31}^j \le C_\gamma m^{\gamma-1} \|h_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} (1+t)^{-\tilde{\alpha}},$$

and, by (10) and (12), one has

$$\overline{\sum} L_{32}^j \le C_\gamma m^{d+\gamma} \|h\| (1+t)^{-\tilde{\alpha}}.$$

L_{33}^j is the hardest term to estimate. We divide it into three cases, in terms of magnitude of velocity variables. First, if $|v| \geq N$, applying (11) two times, it holds that

$$
\begin{aligned}
&\overline{\sum} \int_0^t \int_0^s \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\nu(v)(t-s)} e^{-\nu(v')(s-\tau)} |\ell^c_{\beta'}(v,v') \ell^c_{\beta'}(v',v'')| \\
&\qquad\qquad\qquad\qquad\qquad \times \|\Delta_j h(\tau)\|_{L^\infty_v L^2_x} dv'' dv' d\tau ds \\
&\leq C_\gamma m^{\gamma-1} \|\!|h|\!\| \int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\nu(v)(t-s)} e^{-\nu(v')(s-\tau)} (1+\tau)^{-\tilde{\alpha}} \\
&\qquad\qquad\qquad\qquad\qquad \times \frac{\nu(v')}{1+|v'|^2} |\ell^c_{\beta'}(v,v')| dv' d\tau ds \\
&\leq C_\gamma m^{\gamma-1} \|\!|h|\!\| \int_0^t \int_0^s e^{-\nu(v)(t-s)} (1+s-\tau)^{-1+2/\gamma} (1+\tau)^{-\tilde{\alpha}} \frac{\nu(v)}{1+|v|^2} d\tau ds \\
&\leq C_\gamma m^{\gamma-1} N^{-\delta|\gamma|} \|\!|h|\!\| \int_0^t \int_0^s (1+t-s)^{-1+2/\gamma+\delta} (1+s-\tau)^{-1+2/\gamma} (1+\tau)^{-\tilde{\alpha}} d\tau ds \\
&\leq C_\gamma m^{\gamma-1} N^{-\delta|\gamma|} \|\!|h|\!\| (1+t)^{-\tilde{\alpha}}.
\end{aligned}
$$

Here, $\delta > 0$ is a suitably small constant such that both $0 < \tilde{\alpha} \leq 1 - 2/\gamma - \delta$ and $1 - 2/\gamma - \delta > 1$ hold true. Notice that such a constant $\delta > 0$ exists by the assumption $0 < \beta < |\gamma| + 2$.

The second case is to consider either $\{|v| \leq N,\ |v'| \geq 2N\}$ or $\{|v'| \leq 2N,\ |v''| \geq 3N\}$. For simplicity we only consider the former one since the proof for the latter one is almost the same. Recall that

$$
\begin{aligned}
|\ell^c_{\beta'}(v,v')| &\leq C e^{-N^2/20} |\ell^c_{\beta'}(v,v')| e^{|v-v'|^2/20}, \\
\int_{\mathbb{R}^d} |\ell^c_{\beta'}(v,v')| e^{|v-v'|^2/20} dv' &\leq C_m \frac{\nu(v)}{1+|v|^2},
\end{aligned}
$$

where the second estimate has been shown in [6]. Therefore, $\sum 2^{js} L^j_{33}$ is bounded by

$$
\begin{aligned}
&C_\gamma m^{\gamma-1} \overline{\sum} \int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\nu(v)(t-s)} e^{-\nu(v')(s-\tau)} |\ell^c_{\beta'}(v,v')| \\
&\qquad\qquad\qquad\qquad\qquad \times \frac{\nu(v')}{1+|v'|^2} \|\Delta_j h(\tau)\|_{L^\infty_v L^2_x} dv' d\tau ds \\
&\leq C_\gamma m^{\gamma-1} \|\!|h|\!\| \int_0^t \int_0^s e^{-\nu(v)(t-s)} e^{-N^2/20} \frac{\nu(v)}{1+|v|^2} (1+s-\tau)^{-1+2/\gamma} (1+\tau)^{-\tilde{\alpha}} d\tau ds \\
&\leq C_\gamma m^{\gamma-1} e^{-N^2/20} \|\!|h|\!\| (1+t)^{-\tilde{\alpha}}.
\end{aligned}
$$

Third, if $|v| \leq N$, $|v'| \leq 2N$, and $|v''| \leq 3N$, then we take a small constant $\lambda > 0$ to be chosen later. We divide the τ-integration into two parts $\int_0^s = \int_{s-\lambda}^s + \int_0^{s-\lambda}$. For the first integral $\int_{s-\lambda}^s$, we notice

$$
\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)} (1+\tau)^{-\tilde{\alpha}} d\tau \leq C\lambda (1+s)^{-\tilde{\alpha}},
$$

where C is independent of λ. Therefore, $\sum 2^{js}L_{33}^j$ is correspondingly dominated by

$$
\begin{aligned}
&C\lambda\|h\|\int_0^t\int_{|v'|\le 2N,|v''|\le 3N}(1+s)^{-\tilde{\alpha}}e^{-\nu(v)(t-s)}|\ell^c_{\beta'}(v,v')\ell^c_{\beta'}(v',v'')|dv''dv'ds\\
&\le C_\gamma m^{2-2\gamma}\lambda\|h\|\int_0^t e^{-\nu(v)(t-s)}\Big(\frac{\nu(v)}{1+|v|^2}\Big)^2(1+s)^{-\tilde{\alpha}}ds\\
&\le C_\gamma m^{2-2\gamma}\lambda\|h\|\int_0^t(1+t-s)^{-2(1-2/\gamma)}(1+s)^{-\tilde{\alpha}}ds\\
&\le C_\gamma m^{2-2\gamma}\lambda\|h\|(1+t)^{\beta/\gamma},
\end{aligned}
$$

where the estimate (11) has been used twice in the first inequality. For the second integral $\int_0^{s-\lambda}$, we notice that one can take $\tilde{\ell}_{\beta',N}\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ satisfying

$$
\sup_{|p|\le 3N}\int_{|v'|\le 3N}|\ell^c_{\beta'}(p,v')-\tilde{\ell}_{\beta',N}(p,v')|dv'\le C_mN^{\gamma-1}.
$$

With this approximation function, we decompose the product $\ell^c_{\beta'}(v,v')\ell^c_{\beta'}(v',v'')$ into

$$
\begin{aligned}
\ell^c_{\beta'}(v,v')\ell^c_{\beta'}(v',v'')&=(\ell^c_{\beta'}(v,v')-\tilde{\ell}_{\beta',N}(v,v'))\ell^c_{\beta'}(v',v'')\\
&\quad+(\ell^c_{\beta'}(v',v'')-\tilde{\ell}_{\beta',N}(v',v''))\tilde{\ell}_{\beta',N}(v,v')\\
&\quad+\tilde{\ell}_{\beta',N}(v,v')\tilde{\ell}_{\beta',N}(v',v'').
\end{aligned}
$$

By difference they cancel the singularities and this decompotion suffices to the bounds. Indeed, the integral with the kernel $(\ell_{\beta'}(v,v')-\tilde{\ell}_{\beta',N}(v,v'))\ell_{\beta'}(v',v'')$ is bounded by

$$
\begin{aligned}
&\|h\|\int_0^t\int_0^{s-\lambda}\int_{|v'|\le 2N,|v''|\le 3N}(1+\tau)^{-\tilde{\alpha}}e^{-\nu(v)(t-s)}e^{-\nu(v')(s-\tau)}\\
&\qquad\times|\ell^c_{\beta'}(v,v')-\tilde{\ell}_{\beta',N}(v,v')||\ell_{\beta'}(v',v'')|dv''dv'd\tau ds\\
&\le C_\gamma m^{\gamma-1}\|h\|\int_0^t\int_0^{s-\lambda}\int_{|v'|\le 2N}(1+\tau)^{-\tilde{\alpha}}e^{-\nu(v)(t-s)}e^{-\nu(v')(s-\tau)}\\
&\qquad\times\frac{\nu(v')}{1+|v'|^2}|\ell^c_{\beta'}(v,v')-\tilde{\ell}_{\beta',N}(v,v')|dv'd\tau ds\\
&\le C_\gamma m^{\gamma-1}N^{\gamma-1}\|h\|\int_0^t\int_0^{s-\lambda}e^{-N^\gamma(t-s)}(1+s)^{-\tilde{\alpha}}(1+s-\tau)^{-1+2/\gamma}d\tau ds\\
&\le C_\gamma m^{\gamma-1}N^{-1}\|h\|(1+t)^{-\tilde{\alpha}},
\end{aligned}
$$

where we have used the fact that $\nu(v)\ge cN^\gamma$ if $|v|\le N$. The estimate on the second term is similar and simpler, because $\tilde{\ell}_{\beta',N}(v,v')$ is not singular. Also, in terms of boundedness of $\tilde{\ell}_{\beta',N}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&C_N \overline{\sum} \int_0^t \int_0^{s-\lambda} \int_{|v'|\le 2N, |v''|\le 3N} e^{-\nu(v)(t-s)} e^{-\nu(v')(s-\tau)} \\
&\qquad\qquad \times |\tilde{\ell}_{\beta',N}(v,v')\tilde{\ell}_{\beta',N}(v',v'')| \|\Delta_j h(\tau,v'')\|_{L^2_x} dv'' dv' d\tau ds \\
&\le C_N \overline{\sum} \int_0^t \int_0^{s-\lambda} e^{-cN^\gamma(t-s)} e^{-cN^\gamma(s-\tau)} \|\Delta_j f(\tau)\|_{L^2_{x,v}} d\tau ds \\
&\le C_N \|\!|f|\!\|_{\tilde{\alpha}, \tilde{L}^2_v(B^s_{2,1})} \int_0^t \int_0^{s-\lambda} e^{-cN^\gamma(t-s)} e^{-cN^\gamma(s-\tau)} (1+\tau)^{-\tilde{\alpha}} d\tau ds \\
&\le C_{\gamma,N} \|\!|f|\!\|_{\tilde{\alpha}, \tilde{L}^2_v(B^s_{2,1})} (1+t)^{-\tilde{\alpha}}.
\end{aligned}$$

Here, once again we have used the fact that $\nu(v)$, $\nu(v') \ge cN^\gamma$ if $|v| \le N$ and $|v'| \le 2N$. Also, boundedness of the integral domain has reduced the L^2_v-estimate of $h(v) = \langle v \rangle^{\beta'} f(v)$ to that of $f(v)$.

Finally, summing up all the above estimates, we obtain

$$\begin{aligned}
\|\Delta_j h(t,v)\|_{L^2_x} &\le C(1+t)^{-\tilde{\alpha}} \big(\|h_0\|_{\tilde{L}^\infty_0(B^s_{2,1})} + \|\!|f|\!\|_{\tilde{\alpha}, \tilde{L}^2_v(B^s_{2,1})} \big) \\
&\quad + C'(1+t)^{-\tilde{\alpha}} \big(m^{2-2\gamma}\lambda + m^{\gamma-1}(N^{-1} + e^{-N^2/20} + N^{-\delta|\gamma|}) + m^{d+\gamma} \big) \|\!|h|\!\|,
\end{aligned}$$

where $C = C(\gamma, m, N) > 0$ and $C' > 0$ is independent of (γ, m, N). Now, by taking first $m > 0$ small, next $\lambda > 0$ sufficiently small, and then $N > 0$ sufficiently large, we then derive the desired estimate (7). This completes the proof of Lemma 3.2. $\square$

We note that, on the very last point of the above proof, we first make the term containing $K^m_{\beta'}$, which has a singularity near the origin, small, and then the others, concerned with the term $K^c_{\beta'}$, smaller and larger. This is the reason why K^m is called the small part and K^c the compact part in [8].

Combining Lemma 3.1 and Lemma 3.2 immediately yields the following:

Corollary 3.3. *Let $q \in [1,2]$, $\beta \ge 0$, and $\alpha = 3/2(1/q - 1/2)$. Then the solution $f(t,x,v)$ to the linearized Cauchy problem* (4) *with initial data $f_0(x,v)$ satisfies*

$$\|f\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} \le C(1+t)^{-\alpha} \Big(\|f_0\|_{\tilde{L}^\infty_{\alpha|\gamma|+\beta}(B^s_{2,1})} + \|\nu^{-\alpha_+} f_0\|_{\tilde{L}^2_v(B^s_{2,1})} + \|\nu^{-\alpha_+} f_0\|_{L^2_v L^q_x} \Big).$$

We shall apply the preceding statements for the linear problem to the nonlinear one.

Theorem 3.4. *Assume $s \ge 3/2$, $q = 1$, $\beta \ge 0$ and $\beta > (1 - \alpha/2)\gamma + 3/2 = \gamma/4 + 3/2$. Then the solution $f(t,x,v)$ to the mild form of the Cauchy problem on the nonlinear Boltzmann equation*

$$f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \Gamma(f,f)(s) ds,$$

where e^{tB} is a semigroup generated by the linear part of the Boltzmann equation, enjoys the following estimate:

$$\begin{aligned}
\|f(t)\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} &\le C(1+t)^{-3/4} \|f_0\|_{\tilde{L}^\infty_{(\beta+3|\gamma|/4)}(B^s_{2,1}) \cap \tilde{L}^2_{(3|\gamma|/4)_+}(B^s_{2,1}) \cap L^2_{((3|\gamma|//4)_+} L^1_x} \\
&\quad + C(1+t)^{-3/4} \|\!|f|\!\|^2_{3/4, \tilde{L}^\infty_\beta(B^s_{2,1})}. \qquad (13)
\end{aligned}$$

Outline of proof. Owing to Corollary 3.3, $\|e^{tB}f_0\|_{\tilde{L}^\infty_\beta(B^s_{2,1})}$ can be bounded by the first term on the right-hand side of (13). Thus it suffices to consider the estimate of

$$\int_0^t (1+t-s)^{-3/4}\|\Gamma(f,f)(s)\|_{\tilde{L}^\infty_\beta(B^s_{2,1})}ds. \tag{14}$$

First, we claim that for $s \geq 3/2$ and $(\beta_1, \beta_2) \in \mathbb{R}^2$ with $\gamma + \beta_1 \leq \beta_2$, it holds that

$$\|\Gamma(f,g)\|_{\tilde{L}^\infty_{\beta_1}(B^s_{2,1})} \leq C\|f\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}\|g\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}. \tag{15}$$

To prove this, we introduce the Bony decomposition followed by decomposition of the nonlinear part Γ. For f and $g \in \mathcal{S}'(\mathbb{R}^3_x)$, the Bony decomposition is

$$fg = T_f(g) + T_g(f) + R(f,g),$$

where

$$T_f(g) = \sum_j S_{j-1}(f)\Delta_j g, T_g(f) = \sum_j S_{j-1}(g)\Delta_j f, R(f,g) = \sum_j \sum_{|j-j'|\leq 1} \Delta_{j'} f \Delta_j g.$$

Since Γ is a bilinear form in the x-variable, according to this decomposition, $\Gamma(f,g)$ can be decomposed into the six parts, namely,

$$\Gamma(f,g) = \sum_{k=1}^3 \Gamma^k_{\text{gain}}(f,g) - \Gamma^k_{\text{loss}}(f,g),$$

where

$$\Gamma^1(f,g) = \sum_{j\geq -1} \Gamma(S_j f, \Delta_j g), \Gamma^2(f,g) = \sum_{j\geq -1} \Gamma(\Delta_j f, S_j g),$$
$$\Gamma^3(f,g) = \sum_{j\geq -1}\sum_{|j-j'|\leq 1} \Gamma(\Delta'_j f, \Delta_j g).$$

Advantage of the decomposition is that, when we apply Δ_i in definition of the Besov norms, range of i is restricted according to each j by definition of the cutoff functions χ and ϕ. Term-by-term estimates work by this reason. Establish the estimates term by term, we can obtain (15). Here we only give the estimate of the term involving $\Gamma^1_{\text{loss}}(f,g)$ for brevity. It holds that

$$\begin{aligned}
&\|\Gamma^1_{\text{loss}}(f,g)\|_{\tilde{L}^\infty_{\beta_1}(B^s_{2,1})} \\
&\leq C\overline{\sum}\sum_{|i-j|\leq 4} \sup_v \langle v\rangle^{\beta_1}\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|v-v_*|^\gamma M_*^{1/2}\Delta_j(\Delta_i f_* S_{i-1}g)dv_*\Big)^2dx\Big)^{1/2} \\
&\leq C\overline{\sum}\sum_{|i-j|\leq 4} \sup_v \langle v\rangle^{\beta_1}\int_{\mathbb{R}^d}|v-v_*|^\gamma M_*^{1/2}\|\Delta_i f_* S_{i-1}g\|_{L^2_x}dv_* \\
&\leq C\|f\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}\|g\|_{L^\infty_{\beta_2}L^\infty_x}\sup_v\langle v\rangle^{\beta_1}\int_{\mathbb{R}^d}|v-v_*|^\gamma M_*^{1/2}\langle v_*\rangle^{-\beta_2}\langle v\rangle^{-\beta_2}dv_* \\
&\leq C\|f\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}\|g\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}\sup_v\langle v\rangle^{\beta_1-\beta_2+\gamma}.
\end{aligned}$$

Here, the supremum in the last line is finite thanks to $\gamma+\beta_1 \le \beta_2$.

Second, by the same strategy, we can also prove that for $s \ge 3/2$, $\beta_1 \in \mathbb{R}$, and $\beta_2 \ge 0$ with $\gamma + 3/2 + \beta_1 < \beta_2$, it holds that

$$\|\Gamma(f,g)\|_{\tilde{L}^2_{\beta_1}(B^s_{2,1})} \le C\|f\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}\|g\|_{\tilde{L}^\infty_{\beta_2}(B^s_{2,1})}. \tag{16}$$

We remark that, instead of boundedness of $\sup_v \langle v\rangle^{\beta_1-\beta_2+\gamma}$ above, we require that of

$$\int_{\mathbb{R}^d} \langle v\rangle^{2(\beta_1-\beta_2+\gamma)}dv,$$

and this is the reason we assume the condition $\gamma+3/2+\beta_1<\beta_2$.

Third, as for showing (16), one has

$$\begin{aligned}\|\Gamma(F,G)\|_{L^2_{\beta_1}} &\le \|\Gamma_{gain}(F,G)\|_{L^2_{\beta_1}} + \|\Gamma_{loss}(F,G)\|_{L^2_{\beta_1}}\\ &\le C\|F\|_{L^\infty_{\beta_2}}\|G\|_{L^\infty_{\beta_2}}\end{aligned} \tag{17}$$

for $\gamma+3/2+\beta_1<\beta_2$ with $\beta_1 \in \mathbb{R}$ and $\beta_2 \ge 0$.

Now, applying Corollary 3.3, (14) is bounded by

$$\begin{aligned}C\int_0^t (1+t-s)^{-3/4}\Big(&\|\Gamma(f,f)(s)\|_{\tilde{L}^\infty_\beta(B^s_{2,1})} + \|\nu^{-3/4}\Gamma(f,f)(s)\|_{\tilde{L}^2_v(B^s_{2,1})}\\ &+ \|\nu^{-3/4}\Gamma(f,f)(s)\|_{L^2_vL^1_x}\Big)ds.\end{aligned}$$

Applying estimates (15), (16), and (17) with suitable choices of β_1 and β_2, we obtain the desired estimate. $\square$

The theorem above provides the global a priori estimates stated in the following

Corollary 3.5. *Assume $-3<\gamma<0$, $q=1$, $s\ge 3/2$, $\beta \ge 0$ and $\beta > (1-\alpha/2)\gamma + 3/2 = \gamma/4+3/2$. Then there exist $\varepsilon>0$ and $C>0$ such that if*

$$\|f_0\|_{\tilde{L}^\infty_{(\beta+3|\gamma|/4)}(B^s_{2,1})\cap\tilde{L}^2_{(3|\gamma|/4)_+}(B^s_{2,1})\cap L^2_{((3|\gamma|/4)_+}L^1_x} \le \varepsilon,$$

then the solution $f(t,x,v)$ to the Boltzmann equation with initial datum $f_0(x,v)$ satisfies

$$|\!|\!| f |\!|\!|_{3/4,\tilde{L}^\infty_\beta(B^s_{2,1})} \le C\|f_0\|_{\tilde{L}^\infty_{(\beta+3|\gamma|/4)}(B^s_{2,1})\cap\tilde{L}^2_{(3|\gamma|/4)_+}(B^s_{2,1})\cap L^2_{((3|\gamma|/4)_+}L^1_x}.$$

Together with the inclusion $L^\infty_{\beta_1} \hookrightarrow L^2_{\beta_2}$ for $\beta_1 > \beta_2+3/2$ and the local-in-time existence, Corollary 3.5 yields Theorem 1.4 with the help of the standard continuity argument. This is outlined proof of Theorem 1.4. $\square$

4 Conclusion

Although well-posedness of (3) has been thoroughly studied for decades, the theory of harmonic analysis shed light on new approaches to the equation. We expect that there are other properties of the equation (not necessarily the perturbation problem) that can be uncovered via such theories; see [4] and [13] for example. For this reason, we believe that the analysis in this direction is a strong tool for the study of the Boltzmann equation.

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